9. GROUP $\left\{ \begin{matrix} \text{SYMMETRY} \\ \text{INVARIANCE} \end{matrix} \right\}$ COVARIANCE MODELS

(S.A. ANDERSSON (Copenhagen) + MDP)

STUDY COVARIANCE STRUCTURES WHICH EXHIBIT SYMMETRIES INDUCED BY A (FINITE) GROUP OF (ORTHOGONAL) LINEAR TRANSF'S.

$\underset{\sim}{X} : p \times 1$ a random vector ($\equiv$ MULTIVARIATE OBSERVATION)

$\Sigma : p \times p = \text{cov}(\underset{\sim}{X})$ ($\equiv$ covariance matrix of errors)

$\mathcal{S}_p^+ = \{$all positive definite symmetric real $p \times p$ matrices$\}$

$$[\mathcal{S}_p, \quad \Sigma \in \mathcal{S}_p^+]$$

$G = \{I, g_1, \dots, g_{k-1}\}$ = finite group of ORTHOGONAL MATRICES

$$\mathcal{S}_G^+ = \{\Sigma \in \mathcal{S}_p^+ \mid g \Sigma g' = \Sigma \text{ for all } g \in G\}$$

$[\mathcal{S}_G$ a real linear subspace of $\mathcal{S}_p]$

INVERSE PROPERTY: $\Sigma \in \mathcal{S}_G^+ \Rightarrow \Sigma^{-1} \in \mathcal{S}_G^+$:

$[g\Sigma = \Sigma g$ all $g \in G \Rightarrow \Sigma^{-1} g = g \Sigma^{-1}$ all $g \in G]$ $\quad (g' = g^{-1})$

(ALSO, $I \in \mathcal{S}_G^+$) $\leftarrow$ NOT NEEDED - CAN ASSUME $g \Sigma_0 g' = \Sigma_0$ FOR A FIXED $\Sigma_0 \neq I_p$

GROUP SYMMETRY COVARIANCE HYPOTHESIS

$$(*) \qquad \boxed{\operatorname{cov}(\underset{\sim}{X}) = \Sigma \in \mathcal{S}_G^+}$$

This restricts the form of Σ.
IF TRUE, MORE ACCURATE INFERENCES CAN BE MADE.
WE WILL SHOW THAT THE MLE OF Σ UNDER $(*)$ IS SIMPLE TO COMPUTE.

ALSO, CAN TEST ONE GROUP-INVARIANT MODEL AGAINST ANOTHER

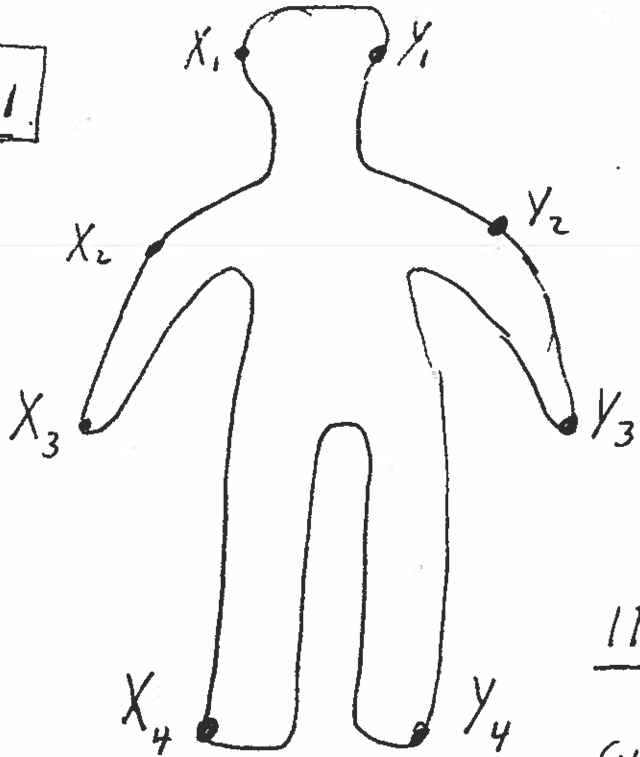
$$H_1 : \Sigma \in \mathcal{S}_{G_1}^+ \quad \text{vs.} \quad H_2 : \Sigma \in \mathcal{S}_{G_2}^+$$

$$\text{where } \mathcal{S}_{G_1}^+ \subseteq \mathcal{S}_{G_2}^+ \quad (G_2 \subseteq G_1)$$

WILL ALSO DISCUSS HOW TO RECOGNIZE WHEN A (LINEAR) RESTRICTION ON Σ ARISES FROM SOME group G. (NOT ALL DO).

FIRST, EXAMPLES.

EX. 1



LEFT-RIGHT SYMMETRY

OBSERVE $\begin{pmatrix} X_1 \\ \vdots \\ X_4 \\ Y_1 \\ \vdots \\ Y_4 \end{pmatrix} \equiv \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix}$

IF ASSUME L-R SYMMETRY:

$$(*) \quad \boxed{\operatorname{cov}\begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix} = \operatorname{cov}\begin{pmatrix} \underset{\sim}{Y} \\ \underset{\sim}{X} \end{pmatrix}}$$

$$\Updownarrow$$

$$(**) \quad \boxed{\begin{aligned} \Sigma_{XX} &= \Sigma_{YY} \\ \Sigma_{XY} &= \Sigma_{YX} \end{aligned}}$$

UNRESTRICTED:

$$\boxed{\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}, \quad \begin{aligned} \Sigma_{XX} &= \Sigma_{XX}' \\ \Sigma_{YY} &= \Sigma_{YY}' \\ \Sigma_{XY} &= \Sigma_{YX}' \end{aligned}}$$

THIS RESTRICTION IS A GROUP SYMMETRY MODEL, FOR

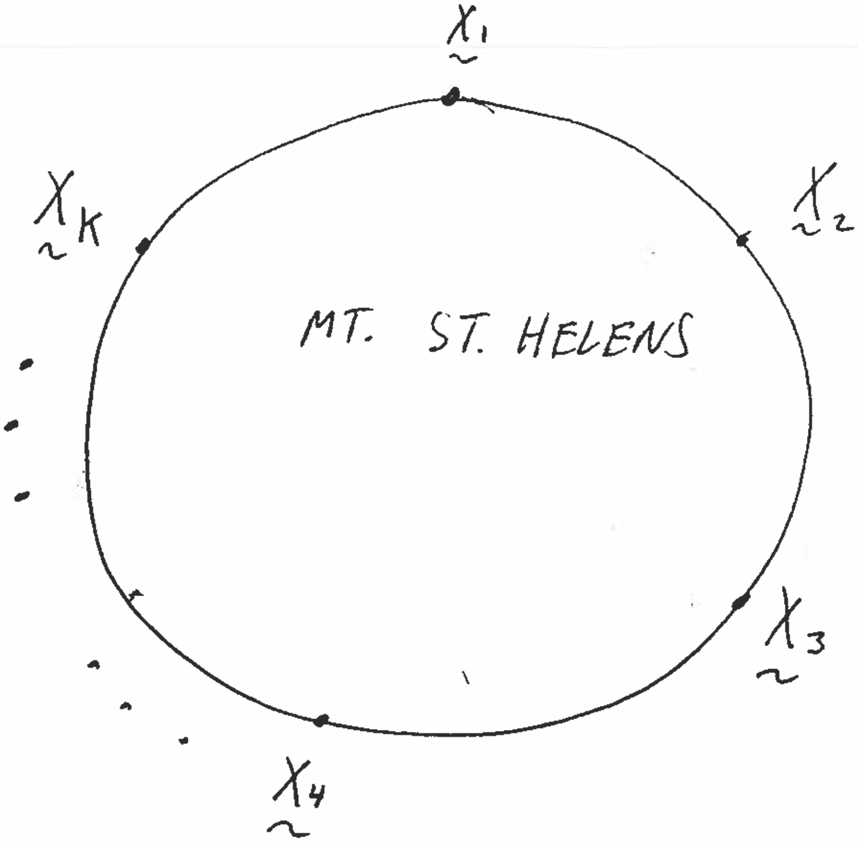
$$\begin{pmatrix} \underset{\sim}{Y} \\ \underset{\sim}{X} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix} \equiv g \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix}$$

$$\& \ (*) \Rightarrow \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix} = g \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix} g' = \begin{pmatrix} \Sigma_{YY} & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_{XX} \end{pmatrix}$$

which is $(**)$.

$$G = \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\}, \quad \mathcal{S}_G = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \middle| \begin{aligned} A &= A' \\ B &= B' \end{aligned} \right\}$$

$\boxed{\text{EX. 2, 3, 4}}$ K BLOCKS OF MULTIVARIATE OBSERVN'S.



$$\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}_1 \\ \vdots \\ \underset{\sim}{X}_K \end{pmatrix}$$

$\underset{\sim}{X}_i : p \times 1$ = (MULTIVARIATE) OBSERVATION AT i^{th} Seismic Station.

$\boxed{\text{Ex. 2}}$ CIRCULAR SYMMETRY:

$$(*) \quad \text{COV}\begin{pmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \\ \vdots \\ \underset{\sim}{X}_K \end{pmatrix} = \text{COV}\begin{pmatrix} \underset{\sim}{X}_2 \\ \underset{\sim}{X}_3 \\ \vdots \\ \underset{\sim}{X}_1 \end{pmatrix} = \text{COV}\begin{pmatrix} \underset{\sim}{X}_3 \\ \underset{\sim}{X}_4 \\ \vdots \\ \underset{\sim}{X}_2 \end{pmatrix} = \cdots$$

$\text{cov}(\underset{\sim}{X}_i, \underset{\sim}{X}_j) = \text{cov}(\underset{\sim}{X}_{i+\delta}, \underset{\sim}{X}_{j+\delta})$ (mod K)

LET $P = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & \vdots \\ \vdots & & \ddots & \ddots & I \\ I & \cdots & & & 0 \end{pmatrix}$

$(*) \iff \operatorname{cov}(\underset{\sim}{X}) = \operatorname{cov}(P^n \underset{\sim}{X}), \quad n = 0, 1, \ldots, K-1$

<u>$n=1$ suffices :</u> [P generates the cyclic gp.]

$$\Sigma \equiv \operatorname{cov}(\underset{\sim}{X}) = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1K} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2K} \\ \vdots & & \ddots & \vdots \\ \Sigma_{K1} & \cdots & & \Sigma_{KK} \end{pmatrix}$$

SATISFIES $\Sigma = P \Sigma P'$ under CIRCULAR SYMMETRY.

<u>$K=3$:</u>

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix} = \begin{pmatrix} 22 & 23 & 21 \\ 32 & 33 & 31 \\ 12 & 13 & 11 \end{pmatrix}$$

$$\iff \begin{cases} \Sigma_{11} = \Sigma_{22} = \Sigma_{33} \equiv A & (A = A') \\ \Sigma_{12} = \Sigma_{23} = \Sigma_{31} = B & (\underline{\text{NOT}}\ B = B') \end{cases}$$

$$\Longleftrightarrow \quad \Sigma = \begin{pmatrix} A & B & B' \\ B' & A & B \\ B & B' & A \end{pmatrix}, \quad \begin{matrix} A = A' \\ (B \neq B') \end{matrix}$$

$\underline{K=4}$:

$$\Sigma = \begin{pmatrix} A & B & C & B' \\ B' & A & B & C \\ C & B' & A & B \\ B & C & B' & A \end{pmatrix} \quad \begin{matrix} A = A' \\ (B \neq B') \\ C = C' \end{matrix}$$

(etc.)

These ARE GROUP SYMMETRY MODELS $\mathcal{S}_G$,

$G = \{I, P, \ldots, P^{K-1}\} \cong \boxed{\text{cyclic group.}}$

HERE do <u>NOT</u> ASSUME $\mathrm{Cov}(\underset{\sim}{X}_i, \underset{\sim}{X}_j) = \mathrm{cov}(\underset{\sim}{X}_j, \underset{\sim}{X}_i)$

IN MULTIVARIATE LITERATURE (OUTSIDE DENMARK),

"BLOCK CIRCULAR SYMMETRY MODEL" HAS REFERRED TO A <u>DIFFERENT</u> MODEL:

EX.3

$k=3$: $\Sigma = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}$ $A=A'$, $B=B'$

$k=4$: $\Sigma = \begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}$ $A=A'$, $B=B'$, $C=C'$

etc.

HERE, <u>do</u> ASSUME $\mathrm{cov}(\underset{\sim}{X}_i, \underset{\sim}{X}_j) = \mathrm{cov}(\underset{\sim}{X}_j, \underset{\sim}{X}_i)$.

(The <u>difference</u> only shows up when the $\underset{\sim}{X}_i$ are <u>multivariate</u>)

This is a group symmetry model $\mathcal{S}_G$

$G = \{I, P, \ldots, P^{k-1}, Q, QP, \ldots, QP^{k-1}\} \cong$ $\boxed{\text{DIHEDRAL GROUP}}$

$Q = \begin{pmatrix} 0 & \cdots & \cdots & I \\ \vdots & & I & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ I & \cdots & \cdots & 0 \end{pmatrix}$, a REFLECTION.

CIRCULAR SYMMETRY OR DIHEDRAL SYMMETRY

EX. 4 COMPLETE BLOCK SYMMETRY:

$$\Sigma = \begin{pmatrix} A & B & \cdots & B \\ B & A & \ddots & \\ \vdots & & \ddots & B \\ B & \cdots & B & A \end{pmatrix} \qquad \begin{matrix} A = A' \\ B = B' \end{matrix}$$

This says
$$\begin{cases} \operatorname{cov}(\underset{\sim}{X}_i) = \operatorname{cov}(\underset{\sim}{X}_j) & \text{all } i, j \\ \operatorname{cov}(\underset{\sim}{X}_i, \underset{\sim}{X}_j) = \operatorname{cov}(\underset{\sim}{X}_\ell, \underset{\sim}{X}_m) & \text{all } i, j, \ell, m \end{cases}$$

$\iff \Sigma = \Pi \Sigma \Pi'$, ALL (BLOCK) PERMUTATION MATRICES Π.

GROUP SYMMETRY MODEL $\mathcal{S}_G$,

$G = \{\text{all (block) permutation mat's}\} \cong$ $\boxed{\text{SYMMETRIC GP. ON } K \text{ LETTERS.}}$

$\boxed{\text{EX. 5}}$ BUT WHAT ABOUT:

$$\Sigma = \begin{pmatrix} A & B \\ B' & A \end{pmatrix} \qquad \begin{matrix} A = A' \\ (B \neq B') \end{matrix}$$

$$\Sigma = \begin{pmatrix} A & B & B \\ B' & A & B \\ B' & B' & A \end{pmatrix} \qquad \begin{matrix} A = A' \\ (B \neq B') \end{matrix}$$

$$\Sigma = \begin{pmatrix} A & B & B & B \\ B' & A & B & B \\ B' & B' & A & B \\ B' & B' & B' & A \end{pmatrix} \qquad \begin{matrix} A = A' \\ (B \neq B') \end{matrix}$$

etc.)

$\boxed{\text{ARE THESE } \mathcal{S}_G^{1}\text{'s FOR SOME } G\text{?}}$

(They ARE linear subspaces of $\mathcal{S}_{pk}$)

(CLOSED UNDER inverses(?) FOR pos def's)

$\boxed{\text{Which linear subspaces are } \mathcal{S}_G^{1}\text{'s?}}$

EX. 6 $\Sigma = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1k} \\ \vdots & & \vdots \\ \Sigma_{k1} & \cdots & \Sigma_{kk} \end{pmatrix}$, $\Sigma_{ij} : p_i \times p_j$

$$G = \left\{ \begin{pmatrix} \pm I_1 & 0 & \cdots & 0 \\ 0 & \pm I_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \pm I_k \end{pmatrix} \right\} \quad (2^k = \# G)$$

$$\mathcal{S}_G = \left\{ \Sigma = \begin{pmatrix} \Sigma_{11} & 0 & \cdots & 0 \\ 0 & \Sigma_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \Sigma_{kk} \end{pmatrix} \right\}$$

A GROUP SYMMETRY MODEL

$\Sigma \in \mathcal{S}_G$ IS CLASSICAL HYPOTHESIS OF INDEPENDENCE OF K SETS OF VARIATES

(See Example 6.7) (& TWA Ch. 9) [NOTE: COMBINE Ex. 6 & Ex. 4 TO GET MODEL: $\Sigma_1 = \cdots = \Sigma_k$ (Assume $p_1 = \cdots = p_k$.]

EX. 7 $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ $\Sigma_{ij} : p \times p$

$$G = \text{group generated by } \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \equiv J$$

$$= \{ I, J, -I, -J \}$$

$$\mathcal{S}_G = \{ \Sigma \mid J \Sigma J' = \Sigma \}$$

$$\underset{J}{\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}} \underset{\Sigma}{\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}} \underset{J'}{\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}} = \begin{pmatrix} \Sigma_{22} & -\Sigma_{21} \\ -\Sigma_{12} & \Sigma_{11} \end{pmatrix}$$

$$\therefore \ \Sigma \in \mathcal{S}_G \iff \left\{ \begin{array}{l} \Sigma_{11} = \Sigma_{22} \ \ (= \Sigma_{11}') \\ \Sigma_{12} = -\Sigma_{21} \ \ (= -\Sigma_{12}') \end{array} \right\}$$

$$\mathcal{S}_G = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\middle|\, \begin{array}{l} A = A' \\ B = -B' \end{array} \right\}$$

$$\xleftrightarrow{1-1} \ A + iB \qquad \underline{\text{COMPLEX HERMITIAN}}$$

HERE, $\Sigma \equiv \operatorname{cov}\begin{pmatrix} \underset{\sim}{X_1} \\ \underset{\sim}{X_2} \end{pmatrix} \in \mathcal{S}_G$

$\iff \underset{\sim}{X_1} + i\underset{\sim}{X_2} \sim$ COMPLEX MULTIVARIATE NORMAL DIST'N

(ASSUMING NORMALITY)

<u>NOTE</u>: Let $\underset{\sim}{Z} = \underset{\sim}{X_1} + i\underset{\sim}{X_2}$. Then $\operatorname{cov}(iZ) = i(\operatorname{cov} Z)i^* = (ii^*)(\operatorname{cov} Z) = \operatorname{cov}(Z)$.

$\therefore$ <u>IF</u> $\underset{\sim}{Z} \sim$ COMPLEX NORMAL (SAY $E\underset{\sim}{Z} = 0$ for simplicity), must <u>require</u> that $\underset{\sim}{Z} \sim i\underset{\sim}{Z}$, that is, $X_1 + iX_2 \sim i(X_1 + iX_2) = (-X_2) + iX_1$,

so require $(X_1, X_2) \sim (-X_2, X_1)$. $\therefore \left\{ \begin{array}{l} \operatorname{cov}(X_1) = \operatorname{cov}(X_2) \equiv A \\ \operatorname{cov}(X_1, X_2) = \operatorname{cov}(-X_2, X_1) \equiv B \end{array} \right\}$ so $\operatorname{cov}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$

EACH OF THESE EXAMPLES OF COVARIANCE MODELS HAVE BEEN ANALYZED SEPARATELY IN THE LITERATURE.

(i.e., MLE'S $\hat{\Sigma}$ FOUND UNDER SYMMETRY RESTRICTION
+ DISTN OF MLE
+ LRT FOR TESTING ONE MODEL VS. ANOTHER
.... etc.)

HOWEVER, DANES AT COPENHAGEN HAVE WORKED OUT THEORY FOR <u>ALL</u> GROUP SYMMETRY MODELS.
(ANDERSSON, BRØNS, JENSEN)

<u>TO ILLUSTRATE</u>: FIND MLE $\hat{\Sigma}$ UNDER RESTRICTION $\Sigma \in \mathcal{S}_G^+$.
(ASSUME NORMALITY)

LET $S \sim \mathcal{W}_p(n, \Sigma)$ (WISHART RAND. MTX)

$$f_\Sigma(S) = \text{CONST.} \frac{|S|^{\frac{n-p-1}{2}}}{|\Sigma|^{\frac{n}{2}}} e^{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}S}$$

$$\log f_\Sigma(S) = \text{CONST}^* - \frac{n}{2}\log|\Sigma| - \frac{1}{2}\operatorname{tr}\Sigma^{-1}S$$

LET $\Lambda = \Sigma^{-1}$

RECALL $\Lambda \in \mathcal{S}_G^+ \Longleftrightarrow \Sigma \in \mathcal{S}_G^+$

$$\therefore \max_{\Lambda \in \mathcal{S}_G^+} \left[\log|\Lambda| - n \operatorname{tr} \Lambda S\right] = ?$$

FACT: $\Lambda \in \mathcal{S}_G^+ \Longrightarrow \operatorname{tr} \Lambda S = \operatorname{tr} g \Lambda g' S \quad (g \in G)$

$$= \operatorname{tr} \Lambda g' S g$$

$$= \operatorname{tr} \Lambda \left[\frac{1}{\#(G)} \sum_{g \in G} g' S g\right]$$

$$\equiv \operatorname{tr} \Lambda S_G$$

FACT: $S_G \in \mathcal{S}_G^+$

$$\therefore \max_{\Lambda \in \mathcal{S}_G^+} \left[\log|\Lambda| - n \operatorname{tr} \Lambda S\right]$$

$$= \max_{\Lambda \in \mathcal{S}_G^+} \left[\log|\Lambda| - n \operatorname{tr} \Lambda S_G\right].$$

$\Downarrow$

$\operatorname{tr}(S - S_G)\Lambda = 0 \quad \forall \Lambda \in \mathcal{S}_G$

$\Downarrow$

$S - S_G \perp \mathcal{S}_G$ ($S_G \in \mathcal{S}_G$)

$\Downarrow$

$S_G = \text{LSE}$

(= ORTHOG. PROJ of S onto $\mathcal{S}_G^+$)

BUT <u>UN</u>RESTRICTED MAX

OCCURS AT $\boxed{\hat{\Lambda} = n S_G^{-1}} \in \mathcal{S}_G^+$

$\therefore$ RESTRICTED MAX OCCURS THERE TOO.

$\therefore$ RESTRICTED MLE $\boxed{\hat{\Sigma} = \frac{1}{n} S_G}$

(DANES): WISHART DIST'N CAN BE DEFINED ON $\mathcal{S}_G^+$ IN NATURAL WAY & S_G FOLLOWS THIS DISTN

<u>FACT</u>: S_G = ORTHOGONAL PROJECTION (≡ LEAST SQUARES PROJ.)

OF S ONTO $\mathcal{S}_G$ (W.R.T. $\langle A, B \rangle = \mathrm{tr}\, AB$)

$\therefore$ (FOR Σ, AS FOR MEANS):

MLE ≡ LEAST SQUARES EST

[GIVES ANOVA-LIKE DECOMPOSITION FOR S when TESTING $\Sigma \in \mathcal{S}_{G_1}$ vs $\mathcal{S}_{G_2}$ - see p. 9.19]

COMPUTATION OF $\overset{1}{\Sigma} \equiv \frac{1}{n} S_G$ EASY

IN EX. 1:

$$S \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \qquad [S_{ij} : p \times p]$$

$$G = \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\}, \quad \mathcal{S}_G = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \middle| \begin{array}{l} A = A' \\ B = B' \end{array} \right\}$$

$$S_G = \frac{\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} + \begin{pmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{pmatrix}}{2} = \left(\begin{array}{c|c} \frac{S_{11}+S_{22}}{2} & \frac{S_{12}+S_{21}}{2} \\ \hline \frac{S_{12}+S_{21}}{2} & \frac{S_{11}+S_{22}}{2} \end{array} \right)$$

IN EX. 2:

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1k} \\ S_{21} & S_{22} & S_{23} & & S_{2k} \\ \vdots & & \ddots & & \vdots \\ S_{k1} & \cdots & & & S_{kk} \end{pmatrix} \qquad [S_{ij} : p \times p]$$

$$G = \{ I, P, P^2, \ldots, P^{k-1} \}, \quad \mathcal{S}_G = \left\{ \begin{pmatrix} A & B & C & \cdots & B' \\ B' & A & B & \ddots & \vdots \\ C' & B' & A & \ddots & C \\ \vdots & & \ddots & \ddots & B \\ B & \cdots & & & A \end{pmatrix} \right\}$$

$$S_G = \frac{1}{K}\sum_{r=0}^{K-1} P^r S(P^r)'$$

$$= \frac{1}{K}\left(\begin{array}{c|c|c|c}
S_{11}+22+\cdots+KK & S_{12}+23+\cdots+K1 & S_{13}+24+\cdots+K2 & \cdots \\
\hline
S_{21}+32+\cdots+1K & \searrow & \searrow & \searrow \\
\hline
\vdots & \searrow & \searrow & \searrow \\
\hline
 & & \searrow & \searrow
\end{array}\right.$$

IN EX. 3:

$$S_G = \frac{1}{2K}\left(\begin{array}{c|c|c|c}
2\begin{pmatrix} S_{11}+22 \\ +\cdots+KK \end{pmatrix} & \begin{array}{l} S_{12}+21 \\ +23+32 \\ +\cdots \\ +K1+1K \end{array} & \begin{array}{l} S_{13}+31 \\ +24+42 \\ +\cdots \\ +K2+2K \end{array} & \cdots \\
\hline
\swarrow & \searrow & \searrow & \searrow \\
\hline
\swarrow & \searrow & \searrow & \searrow \\
\hline
 & & \searrow & \searrow
\end{array}\right.$$

IN EX. 4

$$
S_G = \left(\begin{array}{c|c|c|c|c}
\hat{A} & \hat{B} & \hat{B} & \cdots & \hat{B} \\ \hline
\hat{B} & \hat{A} & \hat{B} & & \hat{B} \\ \hline
\hat{B} & \hat{B} & \hat{A} & & \vdots \\ \hline
\vdots & & & \ddots & \hat{B} \\ \hline
\hat{B} & \cdots & & \hat{B} & \hat{A}
\end{array}\right)
$$

$$\hat{A} = \frac{1}{K}\left[S_{11} + S_{22} + \cdots + S_{KK}\right]$$

$$\hat{B} = \frac{1}{K(K-1)}\left[\sum_{i \neq j} S_{ij}\right]$$

IN EX. 6: $$S_G = \left(\begin{array}{c|c|c|c} S_{11} & 0 & \cdots & 0 \\ \hline 0 & S_{22} & & \vdots \\ \hline \vdots & & \ddots & 0 \\ \hline 0 & \cdots & 0 & S_{KK} \end{array}\right)$$

IN EX. 7: $$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \qquad [S_{ij} : p \times p]$$

$$G = \{I, J\} \cup \{-I, -J\}, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

$$\mathcal{S}_G = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \middle| \begin{array}{l} A = A' \\ B = -B' \end{array} \right\}$$

$$S_G = \frac{\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} + \begin{pmatrix} S_{22} & -S_{21} \\ -S_{12} & S_{11} \end{pmatrix}}{2} = \left(\begin{array}{c|c} \frac{S_{11}+S_{22}}{2} & \frac{S_{12}-S_{21}}{2} \\ \hline \frac{S_{21}-S_{12}}{2} & \frac{S_{11}+S_{22}}{2} \end{array}\right)$$

NOTE: $$\left(\frac{S_{11}+S_{22}}{2}\right) + i\left(\frac{S_{12}-S_{21}}{2}\right)$$

$$\sim \text{COMPLEX WISHART}\left(\Sigma = \left(\frac{\Sigma_{11}+\Sigma_{22}}{2}\right) + i\left(\frac{\Sigma_{12}-\Sigma_{21}}{2}\right)\right)$$

TESTING $\boxed{H_0: \Sigma \in \mathcal{S}^+_{G_0}}$ vs. $\boxed{H: \Sigma \in \mathcal{S}^+_G}$ (BRIEF)

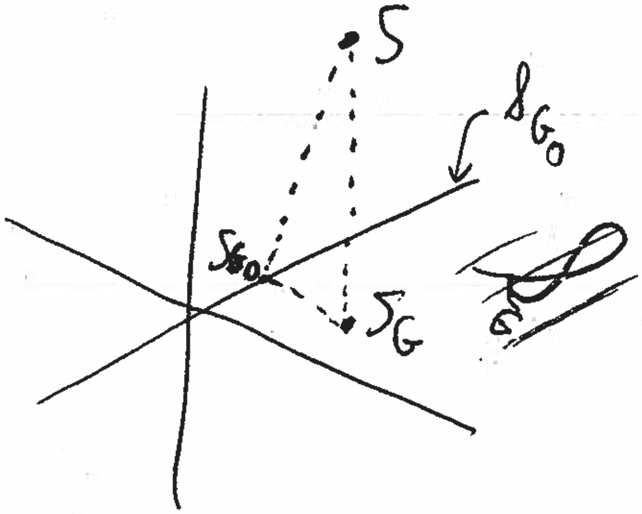
$$\left[\begin{array}{l} G_0 \supseteq G \\ \mathcal{S}^+_{G_0} \subseteq \mathcal{S}^+_G \end{array}\right]$$

"ANOVA"-DECOMPOSITION:

$S \sim W_p(n, \Sigma)$ $(n \geq p)$

$\frac{1}{n} S_G$ = MLE OF Σ under H

= ORTHOG PROJ OF S ONTO $\mathcal{S}_G$



$$\boxed{\begin{aligned} S = \; & S_{G_0} \\ & + (S_G - S_{G_0}) \\ & + (S - S_G) \end{aligned}}$$

"RESIDUAL"

① ANOVA-LIKE. HOWEVER (UNDER NORMALITY) (INVARIANT) TESTS OF H_0 vs. H BASED ON EIGENVALUES OF

$$(S_G - S_{G_0})\, S_{G_0}^{-1}, \quad \text{NOT} \quad (S_G - S_{G_0})(S - S_G)^{\text{"}-1\text{"}}$$

$\Rightarrow$ (NOT ROBUST) (TRY)

② LRT STATISTIC is $\det S_G / \det S_{G_0} \sim \prod$ BETAS UNDER H_0 | EXAMPLE IN §6.

Testing one Group symmetry model against another – DETAILS.

Consider two finite subgroups $G_0 \supseteq G$ of $\mathcal{O}_p = \{p\times p \text{ orthogonal matrices}\}$.

$G_0 \supseteq G \Rightarrow \mathcal{S}_{G_0} \subseteq \mathcal{S}_G$ (G imposes fewer restrictions)

$\therefore$ We can test $H_0: \Sigma \in \mathcal{S}_{G_0}^+$ vs $H: \Sigma \in \mathcal{S}_G^+$, based on $S \sim W_p(n, \Sigma)$. $(n \geq p)$

The LRT statistic is:

$$\lambda = \frac{\max\limits_{\Sigma_0 \in \mathcal{S}_{G_0}^+} \frac{|S|^{\frac{n-p-1}{2}}}{|\Sigma_0|^{n/2}} e^{-\frac{1}{2}\operatorname{tr}\Sigma_0^{-1} \underline{\underline{S_{G_0}}}}}{\max\limits_{\Sigma \in \mathcal{S}_G^+} \frac{|S|^{\frac{n-p-1}{2}}}{|\Sigma|^{n/2}} e^{-\frac{1}{2}\operatorname{tr}\Sigma^{-1} \underline{\underline{S_G}}}}$$

(see p. 13)

$$S_{G_0} = \frac{1}{\#(G_0)} \sum_{g_0 \in G_0} g_0 S g_0'$$

$$S_G = \frac{1}{\#(G)} \sum_{g \in G} g S g'$$

$$= \left(\frac{|\hat\Sigma|}{|\hat\Sigma_0|}\right)^{n/2}$$

(see p. 14)

$$\hat\Sigma_0 = \frac{1}{n} S_{G_0}$$

$$\hat\Sigma = \frac{1}{n} S_G$$

$$= \left(\frac{|S_G|}{|S_{G_0}|}\right)^{n/2}.$$

Note that $0 \leq \lambda \leq 1$. The null distribution (under H_0) of λ can be approximated by a χ^2-distribution: $-2\log\lambda \sim \chi^2_{\dim H - \dim H_0}$. This approximation can be improved by the Box approximation (cf. T.W. Anderson's multivariate book). This requires knowledge of the moments of λ, equivalently, the moments of $\frac{|S_G|}{|S_{G_0}|}$. These moments are obtained by the following interesting identity:

(1). $$E\left(\frac{|S_G|}{|S_{G_0}|}\right)^\alpha = \frac{E|S_G|^\alpha}{E|S_{G_0}|^\alpha} \quad \text{For all } \alpha > 0.$$

Both S_G and S_{G_0} can be shown to have generalized Wishart distributions, so that both $|S_G|$ and $|S_{G_0}|$ are distributed as products of independent (central) chi-square variates under H_0, from which their moments are readily obtained. (See Andersson & Madsen (1998), Ann. Statist, Appendix A.7.) "Symmetry & Lattice Conditional Independence in a Multivariate Normal Distribution".

It remains to verify (i). This requires the following lemma.

Lemma. Under H_0, λ and S_{G_0} are independent. more generally, replace λ by any fn. of $ch(\Sigma_{G_0}^{-1} S_G)$

Proof: Suppose that $\Sigma = \Sigma_0 \in \mathcal{S}_{G_0}^+$. Then (see below) $\Sigma_0^{\frac{1}{2}} \in \mathcal{S}_{G_0}$, hence $\Sigma_0^{-\frac{1}{2}} \in \mathcal{S}_{G_0}^+$. Then

$$\lambda^{2/n} \equiv \frac{|S_G|}{|S_{G_0}|} = \frac{\left|\frac{1}{\#G}\sum_g g S g'\right|}{\left|\frac{1}{\#G_0}\sum_{g_0} g_0 S g_0'\right|} = \frac{\left|\frac{1}{\#G}\sum_g \Sigma_0^{-\frac{1}{2}} g S g' \Sigma_0^{-\frac{1}{2}}\right|}{\left|\frac{1}{\#G_0}\sum_{g_0} \Sigma_0^{-\frac{1}{2}} g_0 S g_0' \Sigma_0^{-\frac{1}{2}}\right|}$$

$$= \frac{\left|\frac{1}{\#G}\sum_g g\left(\Sigma_0^{-\frac{1}{2}} S \Sigma_0^{-\frac{1}{2}}\right) g'\right|}{\left|\frac{1}{\#G_0}\sum_{g_0} g_0\left(\Sigma_0^{-\frac{1}{2}} S \Sigma_0^{-\frac{1}{2}}\right) g_0'\right|} \qquad \left(\begin{array}{l}\text{since } g \Sigma_0^{-\frac{1}{2}} g' = \Sigma_0^{-\frac{1}{2}} \\ \quad\;\; g_0 \Sigma_0^{-\frac{1}{2}} g_0' = \Sigma_0^{-\frac{1}{2}}\end{array}\right)$$

But $\Sigma_0^{-\frac{1}{2}} S \Sigma_0^{-\frac{1}{2}} \sim W_p(n, I)$. Thus the distribution of λ does not depend on $\Sigma_0 \in \mathcal{S}_{G_0}^+$, so λ is an <u>ancillary</u> statistic under H_0.

Next, because the pdf of S under H_0 is given by (pp <u>13</u>-14)

$$f_{\Sigma_0}(S) = \frac{|S|^{\frac{n-p-1}{2}}}{|\Sigma_0|^{n/2}} e^{-\frac{1}{2} \operatorname{tr} \Sigma_0^{-1} \underline{\underline{S_{G_0}}}},$$

a linear exponential family with natural parameter Σ_0^{-1}, it follows from the theory of exponential families that S_{G_0} is a complete and sufficient statistic under H_0. Therefore, by Basu's Lemma (cf. Lehmann, <u>Testing Statistical Hypotheses</u>, Chapter 5), λ and S_{G_0} are independent under H_0.

To see that $\Sigma \in \mathcal{S}_G^+ \Rightarrow \Sigma^{\frac{1}{2}} \in \mathcal{S}_G$, let $\Sigma = \Gamma D_{\underset{\sim}{l}} \Gamma'$ be the spectral decomposition of Σ, so that $\underset{\sim}{l} \equiv (l_1, \cdots, l_p)$ are the eigenvalues of Σ. Let $h(x)$ be any <u>polynomial</u> such that $h(l_i) = l_i^{\frac{1}{2}}$, $i = 1, \cdots, p$. Then

$$\Sigma^{\frac{1}{2}} = \Gamma D_{\underset{\sim}{l}}^{\frac{1}{2}} \Gamma' \equiv \Gamma \begin{pmatrix} l_1^{\frac{1}{2}} & & 0 \\ & \ddots & \\ 0 & & l_p^{\frac{1}{2}} \end{pmatrix} \Gamma' = \Gamma \begin{pmatrix} h(l_1) & & 0 \\ & \ddots & \\ 0 & & h(l_p) \end{pmatrix} \Gamma' = \Gamma h(D_{\underset{\sim}{l}}) \Gamma'$$

$$= h(\Gamma D_{\underset{\sim}{l}} \Gamma') = h(\Sigma). \text{ But } \Sigma \in \mathcal{S}_G \Rightarrow h(\Sigma) \in \mathcal{S}_G. \text{ QED.}$$

(ALL PROOFS, DISTRIBUTION THY, ETC., BASED ON:)

FIRST CHARACTERIZATION OF $\mathcal{S}_G$ (ANDERSSON 1975)

SPLITTING THEOREM: $\mathcal{S}_0$ a linear subspace in $\mathcal{S}_p$.

Then $\mathcal{S}_0 = \mathcal{S}_G$ for some G iff $\exists$ a change of basis ($\equiv$ NONSINGULAR TRANSFORMATION) s.t. $\mathcal{S}_0$ consists of all MATRICES OF THE BLOCK diagonal FORM

$$\begin{bmatrix} \begin{pmatrix} \Sigma_1 & & 0 \\ & \ddots & \\ 0 & & \Sigma_1 \end{pmatrix} & & & \bigcirc \\ & \begin{pmatrix} \Sigma_2 & & 0 \\ & \ddots & \\ 0 & & \Sigma_2 \end{pmatrix} & & \\ & & \ddots & \\ \bigcirc & & & \begin{pmatrix} \Sigma_t & & 0 \\ & \ddots & \\ 0 & & \Sigma_t \end{pmatrix} \end{bmatrix}$$

WHERE:

EACH Σ_i RANGES OVER

$$\mathcal{S}_{p_i} \equiv \{\text{all real symmetric matrices}\} \equiv \mathcal{S}_{\mathbb{R}}^{(p_i)} : p_i \times p_i$$

or

$$\left\{ \begin{array}{l} \text{all real symmetric matrices} \\ \text{of complex structure} \\ \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad \begin{array}{l} A = A^T \\ B = -B^T \end{array} \end{array} \right\} \equiv \mathcal{S}_{\mathbb{C}}^{(q_i)} : 2q_i \times 2q_i$$

or

$$\left\{ \begin{array}{l} \text{all real symmetric matrices} \\ \text{of quaternion structure} \\ \quad (4 \times 4 \text{ BLOCK}) \end{array} \right\} \equiv \mathcal{S}_{\mathbb{Q}}^{(r_i)} : 4r_i \times 4r_i$$

PROOF: (VIA Schur's LEMMA)

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ie.,
$$\begin{aligned} \mathcal{S}_0 = & \left(\mathcal{S}_{\mathbb{R}}^{(p_1)} \otimes I_{m_1}\right) \oplus \cdots \oplus \left(\mathcal{S}_{\mathbb{R}}^{(p_k)} \otimes I_{m_k}\right) \\ & \oplus \left(\mathcal{S}_{\mathbb{C}}^{(q_1)} \otimes I_{m_1}\right) \oplus \cdots \oplus \left(\mathcal{S}_{\mathbb{C}}^{(q_l)} \otimes I_{m_l}\right) \\ & + \left(\mathcal{S}_{\mathbb{Q}}^{(r_i)} \otimes I_{s_1}\right) \oplus \cdots \oplus \left(\mathcal{S}_{\mathbb{Q}}^{(r_m)} \otimes I_{s_m}\right) \end{aligned}$$

THUS, UNDER NORMALITY, A GROUP SYMMETRY MODEL REDUCES TO A PRODUCT OF INDEPENDENT "SIMPLE" MODELS WHERE THE DISTRIBUTION THEORY IS "EASY"

BUT: SPLITTING THM. IS AN EXISTENCE THEOREM. (EXTRINSIC CHAR'N).
DON'T KNOW HOW TO FIND THE APPROPRIATE BASIS TRANSFORMATION, OR IF ONE EXISTS.

IT DOES NOT TELL US HOW TO RECOGNIZE IF A GIVEN SUBSPACE $\mathcal{S}_0$ IS AN $\mathcal{S}_G$ FOR SOME G.

NEED AN INTRINSIC CHARACTERIZATION.
("REFLEXIVE" – see p. 9.27)

CONSIDER SOME EXAMPLES OF $\mathcal{S}_0$'s:

$\mathcal{S}_0$ = ALL (REAL SYMMETRIC) MATRICES OF FOLLOWING FORMS:

		$\mathcal{S}_G$?
(1) $\begin{pmatrix} A & B \\ B' & A \end{pmatrix}$	$A = A'$ $(B \neq B')$	NO NOT "REFLEXIVE" ← (INVERSE FAILS, BUT NOT NECESSARY)

CONSIDER

$\begin{pmatrix} I & B \\ B' & I \end{pmatrix} = S$. Then $S^{-1} = \begin{pmatrix} (I-BB')^{-1} & \times \\ \times & (I-B'B)^{-1} \end{pmatrix}$

$\therefore$ Require $I - BB' = I - B'B$, i.e. $BB' = B'B$

FALSE [illegible]

$\mathcal{S}_0$ | $\mathcal{S}_G$ (?)

(2)

$$\begin{pmatrix} A & B & \cdots & B \\ B^- & A & \ddots & \vdots \\ \vdots & \ddots & A & B \\ B^- & \cdots & B^- & A \end{pmatrix}$$

$A = A^-$

$(B \neq B^-)$

NO

NOT REFLEXIVE

(INVERSE FAILS BUT NOT NECESSARY)

(3)

$$\begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix}$$

a, b, c, d Reals

BETTER Ex

$$\begin{pmatrix} \sigma & a & a & a & a \\ a & \sigma & b & b & b \\ a & b & \sigma & c & c \\ a & b & c & \sigma & d \\ a & b & c & d & \sigma \end{pmatrix}$$

$(I \in \mathcal{S}_0)$ OK

~~NO~~ Yes

$I \notin \mathcal{S}_0$ ← BUT NOT Necess!

(4)

$$\begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

a, b Reals

special case of $\begin{pmatrix} a & b & c \\ b & a & c \\ c & c & a \end{pmatrix}$ — see above

NO?

INVERSE PROPERTY FAILS

$a^{33} \neq a^{11}, a^{22}$

BUT IS THIS REFLEXIVE?

(5)

$$\begin{pmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{pmatrix}$$

a, b Reals

NO?

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix} a^{-1} (0 \ b) \neq \begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix}$$

(INVERSE FAILS)

(6) $\left(\begin{array}{cc|cc} a & b & c_1 & b \\ b & a & b & c_2 \\ \hline c_1 & b & a & b \\ b & c_2 & b & a \end{array}\right)$ a, b, c_1, c_2 REALS

NO?

INVERSE FAILS

(BUT IS THIS REFLEXIVE?)

---

(7) $\begin{pmatrix} a & b & b \\ b & c & d \\ b & d & c \end{pmatrix}$ a, b, c, d REALS

YES

REFLEXIVE

$G = \left\{ \left(\begin{array}{c|c} 1 & 0\ 0 \\ \hline \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} & \Pi \end{array}\right) \middle| \ \Pi \text{ 2x2 PERM MTX} \right\}$

---

(8) $\begin{pmatrix} a & b & b \\ b & c & d \\ b & d & e \end{pmatrix}$ a, b, c, d, e REAL

NO?

INVERSE FAILS

(BUT IS THIS REFLEXIVE?)

SECOND (INTRINSIC) CHARACTERIZATION OF $\mathcal{S}_G$:

(CLASSICAL, IN ANDERSSON (1975), 1-LINE PF):

$$\boxed{\mathcal{S}_0 = \mathcal{S}_G \text{ for some } G \iff \mathcal{S}_0 \text{ REFLEXIVE}}$$

FOR $\mathcal{S} \subseteq \mathcal{S}_p$, $\mathcal{G} \subseteq \mathcal{G}_p$ (= all $p \times p$ NONSING. MATX)

DEFINE:

$$\mathcal{G}(\mathcal{S}) = \{ g \text{ NONSINGULAR MATX} \mid gS = Sg \text{ all } S \in \mathcal{S} \}$$

$$\mathcal{S}(\mathcal{G}) = \{ S \text{ SYMMETRIC MATX} \mid gS = Sg \text{ all } g \in \mathcal{G} \}$$

DEFINITION:

$\mathcal{S}_0$ is REFLEXIVE $\iff$ $\boxed{\mathcal{S}_0 = \mathcal{S}(\mathcal{G}(\mathcal{S}_0))}$

(so $\mathcal{S}_0 = \mathcal{S}_G$ where $G = \mathcal{G}(\mathcal{S}_0)$, (a group!))

NOTE: ALWAYS $\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{G}(\mathcal{S}_0))$

A SHORTCUT:

IF $\mathcal{S}_0 =$ LINEAR SPAN $\{S_1, S_2, \cdots, S_r\}$,

Then $\mathcal{D}(\mathcal{S}_0) = \mathcal{D}(\{S_1, \cdots, S_r\})$

THUS IN EX. ①,

NEED COMPUTE

$$\mathcal{D}\left(\left\{\begin{matrix} A & 0 \\ 0 & A \end{matrix}\right\} \middle| A = A'\right)$$

$$\cap\ \mathcal{D}\left(\left(\begin{matrix} 0 & B \\ B' & 0 \end{matrix}\right) \middle| B \neq B'\right)$$